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Chebyshev Polynomials for Unsteady Aerodynamic Calculations in Aeroservoelasticity

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The aerodynamic forces in the reduced frequency domain have to be approximated in the Laplace domain, in order to study the effects of the control laws on the flexible aircraft structure. This type of approximation of unsteady generalized aerodynamic forces from the frequency domain into the Laplace domain, acting on a Fly-By-Wire aircraft, presents an important challenge in the aeroservoelasticity area. In this paper we present a new method for approximation of the generalized aerodynamic forces, using Chebyshev polynomials and their orthogonality properties. A comparison of results expressed in terms of flutter speeds and frequencies obtained by this new method and by the Padé method is presented here. This new approximation method gives excellent results with respect to Padé method and is validated on the aircraft test model from NASA Dryden Flight Research Center.

Nomenclature

$\boldsymbol{C}$	=	modal damping matrix
c	=	wing chord length
$\boldsymbol{K}$	=	modal elastic stiffness matrix
k	=	reduced frequency
M	=	Mach number
$\boldsymbol{M}$	=	modal inertia or mass matrix
$\boldsymbol{Q}$	=	modal generalized aerodynamic force matrix
$\boldsymbol{Q}_I$	=	imaginary part of modal generalized aerodynamic force matrix
$\boldsymbol{Q}_R$	=	real part of modal generalized aerodynamic force matrix
q	=	nondimensional generalized coordinates (with respect to time t)
q_{dyn}	=	dynamic pressure
T	=	Chebyshev polynomial
V	=	true airspeed
V_E	=	equivalent airspeed
V_0	=	reference true airspeed
η	=	generalized coordinates
ν	=	airspeed ratio
ρ	=	true air density
ρ_0	=	reference air density
Φ	=	modal transformation matrix
ω	=	natural frequency

I. Introduction

AEROSERVOELASTICITY represents the combination of several theories regarding different aspects of aircraft dynamics. Studies of aeroservoelastic interactions on an aircraft are very complex problems to solve, but are essential for an aircraft's certification. Instabilities deriving from adverse interactions between the flexible structure, the aerodynamic forces, and the control laws acting upon it can occur at any time inside the flight envelope. Therefore, it is clear that aeroservoelastic interactions are mainly studied in the research field located at the intersection of the following three disciplines: aerodynamics, aeroelasticity, and servocontrols. One main aspect of aeroservoelasticity is the conversion of the unsteady generalized aerodynamic forces $\boldsymbol{Q}(k, M)$ from the frequency domain into the Laplace domain $\boldsymbol{Q}(s)$, where k is the reduced frequency, M is the Mach number, and s is the Laplace variable. There are basically three classical methods to approximate the unsteady generalized forces by rational functions from the frequency domain to the Laplace domain[1–5]: least squares (LS), matrix Padé (MP), and minimum state (MS). To date, the approximation that yields the smallest order time-domain state-space model is the MS method.[5] All three methods use rational functions in the Padé form.

Several aeroservoelastic analysis software codes have been developed for the aerospace industry. The analog and digital aeroservoelasticity method (ADAM) was developed at the Flight Dynamics Laboratory (FDL). ADAM[6] has been used for the nonaugmented X-29A and for two wind-tunnel models: 1) the FDL model (YF-17) tested in a 4.88-m transonic dynamics tunnel and 2) the forward-swept-wing model mounted in a 1.52-m subsonic wind tunnel. Interaction of structures, aerodynamics, and controls[7] was developed at NASA Langley Research Center and has been used on various flight models such as drone for aeroelastic structure testing aeroelastic research wing (ARW) 1, ARW-2, and DC-10 wind-tunnel flutter models, generic X-wing feasibility studies, analyses of elastic oblique-wing aircraft, active-flexible-wing wind-tunnel test programs, generic hypersonic vehicles and high-speed civil transports. Recently, an aeroelastic code, ZAERO,[8] was developed at Zona Technology, and has been used for aeroservoelastic studies.

The desire for an aeroelastic state-space modeling technique that does not necessitate the addition of lag states to the model has led to the development of FAMUSS.[9] The elements of the state-space matrices are determined by a Pk flutter solution and linear and nonlinear least-square fits of the direct solution of the system's transfer function frequency response. This results in fewer fit equations than the rational function approach. Fitting the transfer function frequency response gives a good indication of the difference between the original system and the equivalent system as sensed by the control system. The pole zero model is used to aid in determining the poles of the system. If the generalized force data are available, then the system poles are obtained directly from them. The linear technique requires that the poles be either input manually or input from the pole zero model. The state-space model generated by FAMUSS is coupled with the model of the flight control system in Matrix_x for an aeroservoelastic analysis.

STARS[10] code was developed at NASA Dryden Flight Research Center (DFRC) and has been applied on various projects at NASA DFRC: X-29A, F-18 High Alpha Research Vehicle/Thrust Vectoring

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Control System, B-52/Pegasus, generic hypersonics, National AeroSpace Plane, SR-71/Hypersonic Launch Vehicle, and high-speed civil transport. The STARS program is an efficient tool for aeroservoelastic interactions studies and has an interface with NASTRAN,[11] a computer program frequently used in the aeronautical industry. All of the preceding codes use two main classical methods for aerodynamic force approximations from the frequency domain (aeroelasticity) into the Laplace domain (aeroservoelasticity): LS and MS. We further present a detailed survey on the other methods existing in the literature.

The aerodynamic forces dependence on s can be written as an irrational function even for simple cases such as two-dimensional potential incompressible flows on an aircraft wing profile. During the 1950s, Theodorsen[12] proved that $\boldsymbol{Q}(s)$ could be expressed by use of Hankel's functions. A few years later, Wagner found the first rational approximation for $\boldsymbol{Q}(s)$. Another approach used the approximation of unsteady aerodynamic forces by Padé polynomials. This approach was based on a fractional approximation of the form $P(s)/R(s)$, where P and R are two polynomials in s, for every term of the unsteady force matrix. In this way, every pole of $R(s)$ showed a new state, called augmented state, in the final linear invariant aeroservoelastic system. In case where the initial square matrix has the N dimension, and where a Padé approximation of M order is used, then $N(N+M)$ augmented states will be introduced.

The number of augmented states was reduced by Roger.[3] In his formulation, only $N \times M$ modes were introduced, where N is the number of initial modes. Roger's method is based on the fact that the aerodynamic lag terms remain the same for each element of the unsteady aerodynamic force matrix. This method is called LS and is used in computer aeroservoelastic codes such as STARS and ADAM.

Another method derived from the LS method was proposed by Vepa.[4] This method uses the same denominators for every column of the aerodynamic matrix $\boldsymbol{Q}$ and is called MP.

Various improvements were done for the two methods just presented: LS and MP. One such type of improvement is that one could impose different conditions (restrictions) to these approximations to pass through certain points. Generally, the approximations are imposed to be exact at zero and at two other chosen points. The first point could be chosen to represent the estimated flutter frequency and the second point to represent the gust frequency. The improved methods have been renamed: extended least-squares (ELS) method[1,13] and extended modified matrix Padé (EMMP) method.[14] Later, Karpel[15] proposed a completely different approach in order to solve the preceding approximations. His goal was to find a linear invariant system in the time domain, and so he decided to integrate this information directly into the equation representing the unsteady aerodynamic force values by adding a term similar to the transfer function of a linear system. Because he wanted to find a linear system of reasonable dimensions, he wrote the approximation under the MS form. The advantage of this method over Roger's method is that it provides an excellent approximation with a smaller number of augmented states.

All of the methods just described allow the approximation of unsteady aerodynamic forces for one Mach number at a time. A valid approximation for a range of Mach numbers would be very useful for military fly-by-wire aircraft, where the Mach number varies rapidly during high-speed maneuvers and where aeroservoelastic interactions are extremely important. Poirion[16,17] constructed an approximation allowing the calculation of the unsteady aerodynamic forces for a range of Mach numbers and for a range of reduced frequencies. He used several MS approximations, obtained for several fixed Mach numbers, and a spline interpolation method for Mach-number dependence. Thus, he obtained formulas that allow the unsteady aerodynamic forces to be computed for any couple (k, M), where k is the reduced frequency and M is the Mach number.

Winther et al.[18] provide an efficient procedure for real-time simulations, which is based on a formulation that eliminates the need for auxiliary state variables to represent the unsteady aerodynamics. The solution includes transformations to a body-axes coordinate system and integration of the structural dynamic equations with the quasi-steady, nonlinear equations of motion.

The approximation methods should simultaneously satisfy two opposed criteria: an excellent (exact) approximation, which can be obtained by increasing the number of lag terms, and a linear invariant system in the time domain of a very small dimension (with the smallest possible number of lag terms). For the time being, there is no method adequately satisfying both criteria. In two recent papers, Cotoi and Botez[19,20] proposed a new approach based on a precise Padé approximation. The authors used four order reduction methods for the last term of the approximation, a term that could be seen as a transfer function of a linear system. The approximation error for this new method is 12–40 times less than for the MS method for the same number of augmented states and is dependent on the choice of the order reduction method. However, this method remains very expensive in terms of computing time.

An alternative approach to rational function aerodynamics and minimum-state aerodynamics is presented by Smith.[21] In this paper the authors provide a technical review of the methods for the generation of a state-space model for a flexible aircraft with frequency-varying aerodynamic forcing. The rational function aerodynamics methods described entail the introduction of aerodynamic, or lag states, to the system for accurate approximation of the generalized aerodynamic forcing. Furthermore, it is shown that the P-transform and FAMUSS methods generate state model approximations of the system directly. Neither FAMUSS nor P-transform requires the introduction of additional lag states, and the accuracy between the methods is about the same. The FAMUSS method includes the option of introducing lag states to the system, whereas the P-transform method does not.

In this paper, we present a new method that uses Chebyshev polynomials to produce approximations for $\boldsymbol{Q}(s)$. The lateral dynamics of a half-aircraft test model (ATM) modeled in STARS manual[10] were used. This means that in this paper, for our comparisons, we have used the data containing only the antisymmetric modes of the ATM. After performing the finite element structural modeling and the doublet-lattice aerodynamic modeling on the ATM in STARS, the unsteady aerodynamic forces $\boldsymbol{Q}$ are calculated as functions of the reduced frequencies k and of the Mach number M. Because $\boldsymbol{Q}(k, M)$ can only be tabulated for a finite set of reduced frequencies k, at a fixed Mach number M, it must be interpolated in the s domain in order to obtain $\boldsymbol{Q}(s)$. In this paper we describe a new interpolation method that uses the Chebyshev polynomials and their properties. Results expressed in terms of flutter speeds and frequencies obtained by use of Chebyshev theory applied to Pk flutter method are compared to same type of results obtained by Padé method on the ATM model.

II. Aircraft Equations of Motion

Flexible aircraft equations of motion, where no external forces are included, can be written in the time domain as follows:

$$\tilde{\boldsymbol{M}}\ddot{\eta} + \tilde{\boldsymbol{C}}\dot{\eta} + \tilde{\boldsymbol{K}}\eta + q_{\mathrm{dyn}} Q(k, M)\eta = 0 \tag{1}$$

where $q_{\mathrm{dyn}} = 0.5\rho V^2$ is the dynamic pressure with ρ as the air density and V as the true airspeed; η is the generalized coordinates variable defined as $\boldsymbol{q} = \Phi\eta$, where $\boldsymbol{q}$ is the displacement vector and Φ is the matrix containing the eigenvectors of the following free-vibration problem:

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{K}\boldsymbol{q} = 0 \tag{2}$$

The following transformations are used in Eq. (1):

$$\tilde{\boldsymbol{M}} = \boldsymbol{\Phi}^T\boldsymbol{M}\boldsymbol{\Phi}, \qquad \tilde{\boldsymbol{C}} = \boldsymbol{\Phi}^T\boldsymbol{C}\boldsymbol{\Phi}, \qquad \tilde{\boldsymbol{K}} = \boldsymbol{\Phi}^T\boldsymbol{K}\boldsymbol{\Phi}$$

$$\boldsymbol{Q}(k, M) = \boldsymbol{\Phi}^T A_e(k)\boldsymbol{\Phi} \tag{3}$$

where $\boldsymbol{M}$, $\boldsymbol{K}$, and $\boldsymbol{C}$ are the generalized mass, stiffness, and damping matrices; k, the reduced frequency, is written as $k = \omega b/V$, where ω is the natural frequency and b is the wing semichord length. $A_e(k)$ is the aerodynamic influence coefficient matrix for a given Mach number M and a set of reduced frequency values k. The Laplace transformation is further applied to Eq. (1), and we obtain

$$[\tilde{\boldsymbol{M}}s^2 + \tilde{\boldsymbol{C}}s + \tilde{\boldsymbol{K}}]\eta(s) + q_{\text{dyn}}\boldsymbol{Q}(s)\eta(s) = 0 \tag{4}$$

The approximation of the unsteady generalized aerodynamic forces is essential for the control analysis of our system. Because $\boldsymbol{Q}(k, M)$ can only be tabulated for a finite set of reduced frequencies, at a fixed Mach number M, these unsteady generalized aerodynamic forces must be interpolated in the s domain in order to obtain $\boldsymbol{Q}(s)$. In this paper we describe an interpolation method using the Chebyshev polynomials and its results. The details of Chebyshev polynomial theory are described in the next chapter.

III. Chebyshev Polynomials Theory

A. Chebyshev Polynomials of the First Kind

These polynomials[22] are a set of orthogonal polynomials defined as the solutions to the Chebyshev equation (10) and are denoted as $T_n(x)$. They are used as an approximation to a least-squares fit and are closely connected with trigonometric multiple-angle equations. Chebyshev polynomials of the first kind are implemented in Mathematica as Chebyshev T $[n, x]$ and are normalized so that $T_n(1) = 1$.

B. Continuous Functions by Use of Chebyshev Polynomials Theory

Any continuous function can be expressed by use of Chebyshev polynomials as follows:

$$f(x) = \frac{1}{2}c_0 + \sum_{j=1}^{\infty} c_j T_j(x) \tag{5}$$

where the Chebyshev polynomials have the following form:

$$T_j(x) = \cos[j \arccos(x)] \tag{6}$$

and the coefficients c_j used in Eq. (5) are expressed as follows:

$$c_j = \frac{2}{\pi}\int_{-1}^{1} \frac{f(x)T_j(x)}{\sqrt{1-x^2}}\,\mathrm{d}x \qquad \text{where} \qquad j = 1, 2, \ldots \tag{7}$$

C. Orthogonality of Chebyshev Polynomials

In our new approximation method for unsteady aerodynamic forces, we have used the Chebyshev polynomials because they have a specific orthogonality property. This interesting property allows us to keep the approximation's error within a predetermined bandwidth and can further be expressed as

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} T_r(x)T_s(x)\,\mathrm{d}x = \begin{cases} 0, & r \neq s \\ \pi, & r = s = 0 \\ \dfrac{\pi}{2}, & r = s \neq 0 \end{cases} \tag{8}$$

D. Recurrence Formulas and the Solution of Chebyshev Polynomials

The following recurrence relationships[22–24] have been used in the Chebyshev polynomials new approximation method:

$$T_0(x) = 1, \qquad T_1(x) = x$$

$$T_{r+1}(x) = 2xT_r(x) - T_{r-1}(x) \tag{9}$$

Next, we impose the following condition to find the Chebyshev polynomials solution:

$$T_r(x) = 0 \tag{10}$$

where r specifies the rank of the Chebyshev polynomial. Equation (10) gives the following solution:

$$x = \cos[(2j+1)\pi/2r] \tag{11}$$

Thus, the expression

$$\tilde{Q}_r(x) = (1/2^{r-1})T_r(x) \tag{12}$$

will oscillate with a minimum maximum (extreme) amplitude within the interval $[-1, 1]$.

Proof:

The condition in Eq. (10) is imposed, and we obtain:

$$\cos[r \arccos(x)] = 0 \tag{13}$$

By use of the variables change

$$y = r \arccos(x) \tag{14}$$

we found the following trivial equation solutions:

$$\cos(y) = 0 \tag{15}$$

so that

$$y = (2j+1)(\pi/2) \qquad j = 0, 1, \ldots \tag{16}$$

The solutions are further calculated by inserting Eq. (16) into Eq. (14):

$$r \arccos(x) = (2j+1)(\pi/2) \Rightarrow \arccos(x) = (2j+1)\pi/2r \tag{17}$$

so that we obtain

$$x = \cos[(2j+1)\pi/2r] \qquad j = 0, 1, \ldots, r-1 \tag{18}$$

E. Extreme Amplitudes

$T_r(x)$ is a function defined by cosines, which lets us conclude that between two solutions of this function we will find an extreme of $|1|$ amplitude exactly in the middle of the interval, specifically at

$$x = \cos(j\pi/r) \qquad j = 0, 1, \ldots, r \tag{19}$$

IV. Chebyshev Approximation Method for Aeroservoelastic Interactions Studies

To develop our approximation method, we used the predefined functions using Chebyshev polynomials expressed in Eq. (6), which have already been implemented in the Maple's kernel, in MATLAB®.

These functions (*chebpade* and *chebyshev*) allowed the construction of a polynomial interpolation for the unsteady generalized aerodynamic forces, acting on the ATM for 14 values of reduced frequencies k and nine values of Mach number. The elements forming the matrices of the unsteady generalized aerodynamic forces calculated by the doublet-lattice Method (DLM) in STARS were denoted by $\boldsymbol{Q}(i, j)$ with $i = 1 \ldots 8$ and $j = 1 \ldots 8$ for the first eight elastic modes.

The approximation by means of this method is obtained using a similar path to the one used for the Padé method. For each element of the unsteady aerodynamic force matrix, we have determined a power series development in the following form, by use of the *chebyshev* function:

$$\boldsymbol{Q}_{ij}(s) = \frac{1}{2}c_0^{(ij)} + \sum_{n=1}^{\infty} c_n^{(ij)} T_n^{(ij)}(s) \tag{20}$$

where

$$c_n^{(ij)} = \frac{2}{\pi}\int_{-1}^{1} \frac{\boldsymbol{Q}_{ij}(s)T_n^{(ij)}(s)}{\sqrt{(1-s^2)}}\,\mathrm{d}s \qquad \text{for every} \qquad n = 0, 1, \ldots$$

Next, by use of the *chebpade* function we have found an approximation by rational fractions in the following form:

$$\hat{\boldsymbol{Q}}_{ij}(s) = \frac{\sum_{n=0}^{M} a_n^{(ij)} T_n^{(ij)}(s)}{1 + \sum_{n=1}^{P} b_n^{(ij)} T_n^{(ij)}(s)} \tag{21}$$

where $M = P + 2$.

This new form is very useful because it integrates the orthogonality properties of Chebyshev polynomials and allows us to vary the degree of the numerator and the denominator in order to obtain a very good approximation.

We have compared the results found by means of our Chebyshev approximation method with the results given by another classical interpolation method such as Padé. These results were expressed in terms of the approximation's normalized error.

The Padé method uses a parameter identification solution in order to determine a polynomial fractional form, which identifies an orthogonal polynomial interpolation. This fractional form is the key aspect of this method, because it allows the order reduction system.

We see in Figs. 1 and 2 that our new approximation method by Chebyshev polynomials gives a smaller approximation error than Padé method. Because of Chebyshev polynomials properties, we were able to impose a bandwidth for the error convergence on the approximation for each element of the unsteady generalized aerodynamic force matrices. Both figures show the total normalized approximation error by Chebyshev and Padé polynomials for real and imaginary unsteady aerodynamic forces.

Figure 1 shows the total normalized approximation error for real and imaginary aerodynamic forces for the [15, 13] model order by use of Padé and Chebyshev approximation methods, and Fig. 2 shows the total approximation error for real and imaginary aerodynamic forces for the [16, 14] model order by use of Padé and Chebyshev approximation methods. For this example, we applied both methods on the ATM generated in STARS code at the Mach number $M = 0.5$ and for 14 reduced frequencies $k = [0.0100\ \ 0.1000\ \ 0.2000\ \ 0.3030\ \ 0.4000\ \ 0.5000\ \ 0.5882\ \ 0.6250\ \ 0.6667\ \ 0.7143\ \ 0.7692\ \ 0.8333\ \ 0.9091\ \ 1.0000]$. Some differences at both ends of the approximation interval can be seen in the figures. The polynomial approximation order in the Chebyshev method represents the maximum rank of the Chebyshev polynomials used to form the numerator M and the denominator P in Eq. (21). In other words, in Eq. (21), an approximation order $[M, P] = [16, 14]$ gives $M = 16$ and $P = 14$, where M is the maximum rank of Chebyshev polynomials at the numerator and P is the maximum rank of Chebyshev polynomials at the denominator. In the same way we defined the approximation order for Padé.

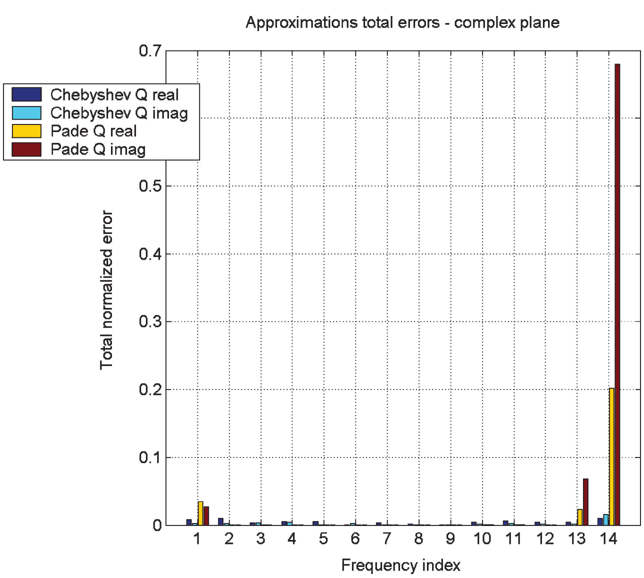

Fig. 1 Total normalized errors Padé and Chebyshev [15, 13] model order, Mach 0.5.

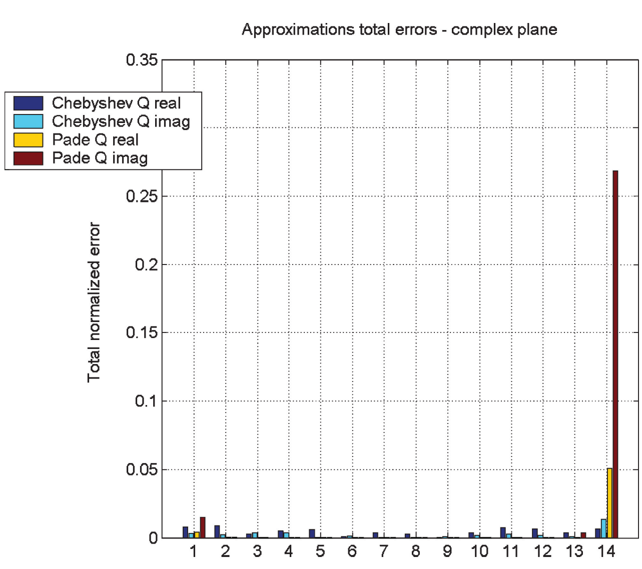

Fig. 2 Total normalized errors Padé and Chebyshev [16, 14] model order, Mach 0.5.

We used only a few different values of the polynomial approximation order by the Padé method and the Chebyshev polynomial fractions method (polynomial order should be equivalent for both methods) in order to calculate the total normalized approximation error, which was found to be much smaller for the Chebyshev polynomial method with respect to the overall approximation error given by Padé.

More specifically, Padé method gives a small error near the approximation point (in our examples, the middle of the reduced frequency interval) and an increased error toward each end of the interval. The Chebyshev approximation method demonstrates an almost constant value of the error all along the approximation interval. The threshold of this error could be imposed from the beginning of the calculations in order to find the unsteady generalized aerodynamic force approximation matrices.

In Figs. 1 and 2, we can only visualize the overall normalized approximation error, calculated for the whole unsteady generalized aerodynamic force matrix at Mach number $M = 0.5$, by use of the two described methods. As exact values for these approximations are difficult to compare on Figs. 1 and 2, we are providing Table 1 to show the numerical values of these errors obtained, for example, at four Mach numbers for three approximation orders ($M = 0.5$ included).

The ATM data imply huge differences between the unsteady generalized aerodynamic force values for different reduced frequencies at different Mach numbers, and the total approximation error given in metrics does not provide a clear understanding of our new

Table 1 Total normalized errors

Mach number	Approximate order	Method	Total normalized error $J_{Q\mathrm{real}}$	$J_{Q\mathrm{imaginary}}$	J_Q
0.4	[16, 14]	Chebyshev	0.0656	0.1670	0.2326
		Padé	0.7831	0.2020	0.9852
	[15, 13]	Chebyshev	0.0643	0.1605	0.2248
		Padé	0.1226	0.5149	0.6376
	[10, 8]	Chebyshev	0.0676	0.1670	0.2346
		Padé	0.0570	0.0705	0.1275
0.5	[16, 14]	Chebyshev	0.0647	0.0343	0.0991
		Padé	0.0552	0.2876	0.3429
	[15, 13]	Chebyshev	0.0672	0.0367	0.1039
		Padé	0.2601	0.7759	1.0361
	[10, 8]	Chebyshev	0.0500	0.0299	0.0799
		Padé	0.1363	0.0556	0.1919
0.6	[16, 14]	Chebyshev	0.1209	0.0911	0.2121
		Padé	0.1451	0.0792	0.2243
	[15, 13]	Chebyshev	0.1209	0.0804	0.2013
		Padé	0.1610	3.7261	3.8871
	[10, 8]	Chebyshev	0.1222	0.0786	0.2009
		Padé	0.3213	0.0289	0.3503
0.7	[16, 14]	Chebyshev	0.0871	0.1710	0.2581
		Padé	1.2893	1.8254	3.1148
	[15, 13]	Chebyshev	0.0867	0.1703	0.2571
		Padé	2.3919	2.8628	5.2547
	[10, 8]	Chebyshev	0.0926	0.1607	0.2533
		Padé	219.2462	5.6375	224.8837

Table 2 Elastic antisymmetric mode definition

Mode number	Mode shape	Acronym
1	Vertical fin first bending	VF1B
2	Fuselage first bending	FUS1B
3	Wing first bending	W1B
4	Wing second bending	W2B
5	Fuselage second bending	FUS2B
6	Wing first torsion	W1T
7	Fin first torsion	F1T
8	Fuselage third bending	FUS3B

method's advantages for all reduced frequencies. For these reasons, we have chosen to normalize this approximation error for each element of the $\boldsymbol{Q}$ matrix, for the real and the imaginary part, at each reduced frequency using the equations:

$$J_{Q\,\mathrm{real}} = \sum_{k=1}^{14}\left[\sum_{i=1}^{N_{\mathrm{modes}}}\left(\sum_{j=1}^{N_{\mathrm{modes}}}\frac{|\boldsymbol{Q}_{ijR\mathrm{new}} - \boldsymbol{Q}_{ijR\mathrm{old}}|}{\sqrt{|\boldsymbol{Q}_{ij}|^2}}\right)\right] * 100\%$$

$$J_{Q\,\mathrm{imaginary}} = \sum_{k=1}^{14}\left[\sum_{i=1}^{N_{\mathrm{modes}}}\left(\sum_{j=1}^{N_{\mathrm{modes}}}\frac{|\boldsymbol{Q}_{ijI\mathrm{new}} - \boldsymbol{Q}_{ijI\mathrm{old}}|}{\sqrt{|\boldsymbol{Q}_{ij}|^2}}\right)\right] * 100\% \tag{22}$$

and

$$J_Q = J_{Q\,\mathrm{real}} + J_{Q\,\mathrm{imaginary}}$$

where $\boldsymbol{Q}_{R\mathrm{old}}$ and $\boldsymbol{Q}_{I\mathrm{old}}$ are the real and the imaginary parts of the unsteady aerodynamic forces given by STARS for the ATM model and $\boldsymbol{Q}_{R\mathrm{new}}$ and $\boldsymbol{Q}_{I\mathrm{new}}$ are the real and the imaginary parts of the unsteady aerodynamic forces approximated by Chebyshev or by Padé theories. N_{modes} is the total number of modes (also the dimension of $\boldsymbol{Q}$), k is the index of the reduced frequency, and J is the total normalized error.

In Table 1, the values presented for Chebyshev and Padé indicate the values of the total normalized errors obtained with these two methods according to Eq. (22). The case of the [10, 8] approximation order for Chebyshev and Padé, at $M = 0.4$, indicates that in this situation Padé approximation gives a smaller error than Chebyshev approximation. We have intentionally introduced this case in Table 1 because it was the only case out of the total of 108 cases that we have studied for the ATM (nine Mach numbers by 12 approximation orders between [6, 4] and [17, 15]), where Padé provided a smaller error than Chebyshev. The difference between the approximation errors of Padé and Chebyshev in this case was however very small (only 0.1%). In Table 1 are presented 12 of the 108 studied cases. The values of the total normalized approximation errors in Table 1 show that, with the $M = 0.4$ and [10, 8] exception, Chebyshev yields the smallest error. Table 2 shows the elastic antisymmetric mode definitions of the ATM model.

V. Open-Loop *Pk* Flutter Analysis

To present the open loop Pk flutter analysis, we need to define the aerodynamic quantities: air density ratio σ, equivalent airspeed V_E, airspeed ratio ν, dynamic pressure q_{dyn}, and reduced frequency k in the following equations:

$$\sigma = \rho/\rho_0 \tag{23}$$

$$V_E = \sqrt{\sigma}V \tag{24}$$

$$\nu = V_E/V_0 \tag{25}$$

$$q_{\mathrm{dyn}} = \tfrac{1}{2}\rho V^2 \tag{26}$$

The formulation for linear aeroelastic analysis in the case of the Pk flutter method is the one presented in Eq. (1), where $\boldsymbol{Q}$, the unsteady generalized aerodynamic force matrix, is usually complex. The real part of $\boldsymbol{Q}$ denoted by $\boldsymbol{Q}_R$ is called the "aerodynamic stiffness" and is in phase with the vibration displacement; the imaginary part of $\boldsymbol{Q}$ denoted by $\boldsymbol{Q}_I$ is called the "aerodynamic damping" and is in phase with the vibration velocity.

This dynamics equation is a second-degree nonlinear equation with respect to the generalized coordinates' variable η. The nonlinearity comes from the fact that the aerodynamic force matrix $\boldsymbol{Q}$ is a function of reduced frequency k, depending of the natural frequency ω as shown:

$$k = \omega c/2V = \omega b/V \tag{27}$$

where b is the semichord. With respect to the dynamics Eq. (1), we associate $\boldsymbol{Q}_R$ with η and $\boldsymbol{Q}_I$ with $\dot{\eta}$. For this reason, as the $\boldsymbol{Q}$ matrix is already a coefficient of η, we need to divide $\boldsymbol{Q}_I$ by ω to express $\boldsymbol{Q}_I$ as a coefficient of $\dot{\eta}$. Thus, aeroelastic Eq. (1) becomes

$$\boldsymbol{M}\ddot{\eta} + [\boldsymbol{C} + (1/\omega)q_{\mathrm{dyn}}\boldsymbol{Q}_I]\dot{\eta} + (\boldsymbol{K} + q_{\mathrm{dyn}}\boldsymbol{Q}_R)\eta = 0 \tag{28}$$

Equation (27) gives ω as a function of k:

$$\omega = 2kV/c \tag{29}$$

We replace q_{dyn} and ω given by Eqs. (26) and (29) in Eq. (28) and obtain the following equation:

$$\boldsymbol{M}\ddot{\eta} + [\boldsymbol{C} + (1/4k)\rho V c\boldsymbol{Q}_I]\dot{\eta} + \left(\boldsymbol{K} + \tfrac{1}{2}\rho V^2\boldsymbol{Q}_R\right)\eta = 0 \tag{30}$$

Equations (23) and (24) give the value of σ:

$$\sigma = V_E^2\big/V^2 = \rho/\rho_0 \tag{31}$$

from where

$$\rho V^2 = \rho_0 V_E^2 \tag{32}$$

If we divide both sides of Eq. (32) by V and we use the definition of σ given by Eq. (24), we obtain

$$\rho V = \rho_0\left(V_E^2\big/V\right) = \rho_0 V_E(V_E/V) = \rho_0 V_E\sqrt{\sigma} \tag{33}$$

Then, we replace Eq. (33) in the coefficient of $\boldsymbol{Q}_I$ and Eq. (32) in the coefficient of $\boldsymbol{Q}_R$ and get

$$\boldsymbol{M}\ddot{\eta} + \left[\boldsymbol{C} + (1/4k)\rho_0 c\sqrt{\sigma}V_E\boldsymbol{Q}_I\right]\dot{\eta} + \left(\boldsymbol{K} + \tfrac{1}{2}\rho_0 V_E^2\boldsymbol{Q}_R\right)\eta = 0 \tag{34}$$

VI. Results and Discussion

To validate our method, we used the antisymmetric modes of the ATM developed in STARS code by the NASA Dryden Flight Research Center. This model includes aerostructural elements (flexible aircraft) and control surfaces (aircraft commands). The aircraft configuration and mode shapes are presented in the STARS[10] manual. First, a free-vibration analysis was performed in the absence of aerodynamics to obtain the free modes of vibration. Then, we calculate aerodynamic forces in the frequency domain $\boldsymbol{Q}(k)$ by the DLM. We obtained, as first step of validation, the same frequencies and modes of vibration by use of our method as the ones obtained by use of STARS code.

Then, by use of the Pk-Chebyshev method we can find the values for the speeds and frequencies where first flutter point occurs. The results of the simulation are also shown in Figs. 3–7. The same simulation parameters were considered as in the STARS computer program: reference semichord length $b = 0.99$ m; reference air density at sea level $\rho_0 = 1.225\ \mathrm{kg/m^3}$; altitude at sea level $Z = 0$ ft, $M = 0.9$; and reference sound airspeed at sea level $a_0 = 340.294$ m/s.

In Table 3, results are expressed in terms of velocities and frequencies at which the first flutter point occurs. In fact, the fuselage lateral first bending mode is a mode with a very small damping. The results found by means of our Pk method that makes use of the Chebyshev polynomial approximation are compared with the results given by Pk-Padé method and the methods programmed in STARS. We have provided the flutter speed and frequency obtained for four approximation orders for the Chebyshev method and the

Table 3 Speed and frequency for the first flutter point

	Flutter point 1 (F1B)		
Method	Speed, kn	Frequency, rad/s	Computation time, s
STARS-ASE	474.1	77.3	——
STARS-*k*	443.3	77.4	——
STARS-*Pk*	441.7	77.4	——
Pk-Padé [8, 6]	445.5	77.5	122
Pk-Padé [9, 7]	445.5	77.5	134
Pk-Padé [10, 8]	445.8	77.5	144
Pk-Padé [11, 9]	446.0	77.5	151
Pk-Chebyshev [8, 6]	446.5	77.5	40
Pk-Chebyshev [9, 7]	446.6	77.5	47
Pk-Chebyshev [10, 8]	446.6	77.5	53
Pk-Chebyshev [11, 9]	446.6	77.5	58

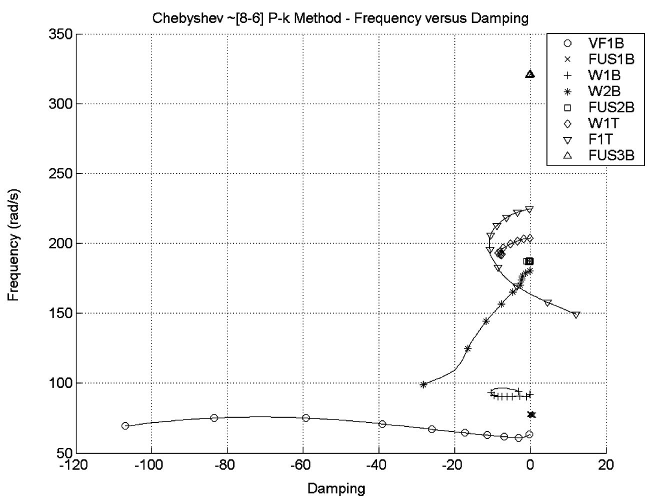

Fig. 3 Frequency vs damping for *Pk*-Chebyshev [8, 6] model order.

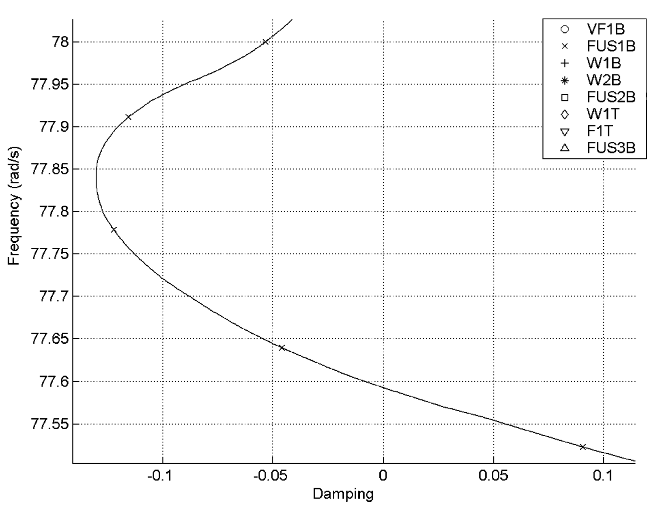

Fig. 4 Frequency vs damping *Pk*-Chebyshev [8, 6] model order—zoomed interest area.

Padé method. The results for speed are given in knots to be able to compare them with the ones provided by STARS, which are also given in knots.

We have found that the results of our method closely match those obtained by the other methods. In addition to that, if we take a closer look at the normalized approximation errors for the unsteady aerodynamic force matrix presented in Table 1 and we also take into account the computation costs of the Chebyshev method and the Padé method we can state that Chebyshev is a better method than Padé method (and consequently, than other methods based on Padé)

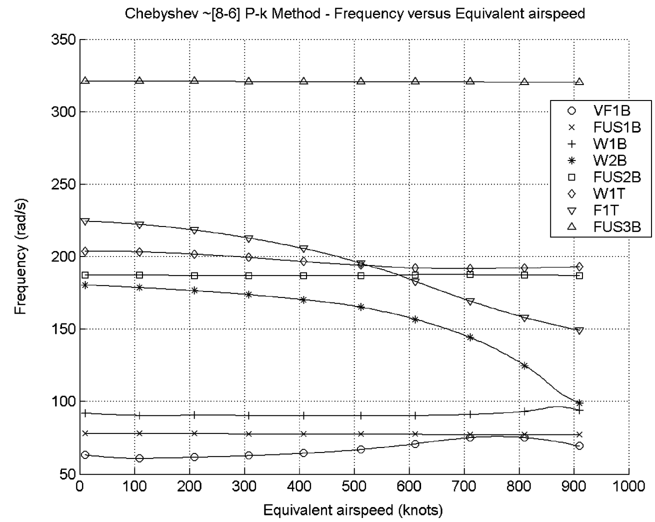

Fig. 5 Frequency vs equivalent airspeed for *Pk*-Chebyshev [8, 6] model order.

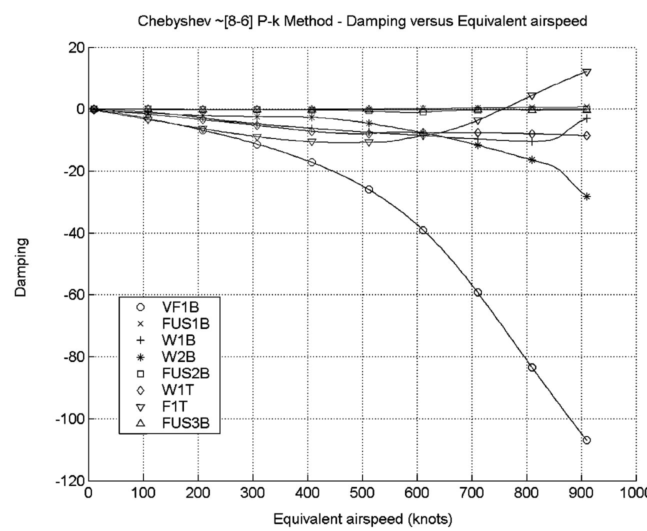

Fig. 6 Damping vs equivalent airspeed for *Pk*-Chebyshev [8, 6] model order.

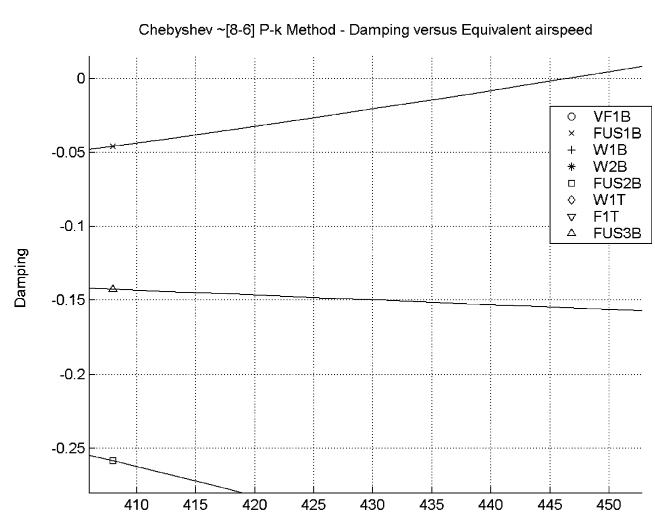

Fig. 7 Damping vs equivalent airspeed *Pk*-Chebyshev [8, 6] model order—zoomed interest area.

and also a more reliable one: using Padé with an approximation order of [10, 8] for Mach number 0.7, we came to the conclusion that this method could sometimes provide unexpectedly large errors. Regarding the computation costs, it takes between 40 s and 1 min to obtain the approximation of the unsteady aerodynamic forces for one Mach number using the Chebyshev method and between 2 min and 2 min and 30 s using the Padé method and 50 MB additional computer memory resources. Using the LS method on the same data, it will take between 20 and 25 min (depending on the number of lags introduced) and about 150 MB more computer memory allocated.

As seen in Table 1, the approximation's normalized total error calculated by the Chebyshev method stabilizes around 0.1, whereas the approximation's normalized error calculated by the Padé method (and consequently by any other method based on Padé rational fractions) presents constant fluctuations dependent upon the approximation order. It has been found that an approximation order of [8, 6] or [9, 7] (see Table 3) would be sufficient in the application of our Chebyshev method on the ATM model to calculate flutter velocities and frequencies. It has also been found that there is a link between the approximation order and the computational costs: the higher the approximation order is, the longer time it takes for computation. The reader should not overlook the fact that the accuracy normally obtained using an increased approximation order could be overweighted by the computational truncation errors. Each one should choose the best-suited approximation order for the data to be computed. Our method tries to find the best compromise between the computational costs and the accuracy of the results.

VII. Conclusions

The Chebyshev approximation method provides an excellent approximation by comparison with the Padé method. However, because the Chebyshev polynomials had to be generated using the data provided on the aircraft test model, which involves quite large differences between the values of the elements contained in the unsteady generalized aerodynamic force matrices ($1e+10$), some restraints regarding the threshold of the approximation error had to be imposed, that is, for smaller elements we have imposed a maximum error value of $1e-4$ and for larger elements a maximum error value of $1e-2$. Without these restraints, the Chebyshev polynomials cannot be generated.

We could see that using the Chebyshev method in open loop we were able to find excellent approximated values for the flutter speed and the frequencies at which flutter occurs. The results in Table 3 were presented to show that a rather small approximation order would be sufficient for the Chebyshev method to provide accurate information for flutter speed and frequency, and a smaller approximation order means less computation time. One of the most important achievements of our new method, if not the most important, is the fact that the computation time for the open-loop case is up to three times smaller than in the Pk-Padé method and up to 30 times smaller than in the least-squares case, even for an increased approximation order.

Generally, an approximation method is considered to be better if it satisfies the following criteria: it provides a smaller approximation error; it is more rapid; and, ultimately, uses less computer resources, which could be in some cases a crucial criterion (when a significantly larger quantity of data to be approximated is used). In this case, Chebyshev provides better results, satisfying all criteria.

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